AN EXACT TEST FOR RENEWAL INCREASING MEAN RESIDUAL LIFE

Sudheesh K Kattumannil

Indian Statistical Institute, Chennai, India

Abstract. In this paper, we develop an exact test for testing exponentiality against renewal increasing mean residual life class. Pitman's asymptotic efficacy value shows that our test perform well. Some numerical results are presented to demonstrate the performance of the testing method. We also discuss how the proposed method incorporates the right censored observations.

Keywords: Exponential distribution; Renewal increasing mean residual life; Replacement model; Shock Model; U-statistics.

1. Introduction

When a device is experiencing random number of shocks governed by a homogeneous Poisson process, the concept of renewal increasing mean residual life is very much useful to study the properties of age replacement model. In this context, test for exponentiality against renewal increasing mean residual life class is used to determine whether to adopt a planned replace model over unscheduled one. Sepehrifar et al. (2015) developed a non-parametric test against RIMRL_{shock} class and obtained a critical region based on the asymptotic theory of U-statistics. We noted that the critical region developed by Sepehrifar et al. (2015) is incorrect (see Remark 2.1). Motivated by Sepehrifar et

al. (2015), we develop an exact test for testing exponentiality against RIMRL$_{shock}$ class. We also obtain the correct critical region of the asymptotic test proposed by Sepehrifar et al. (2015). The case with censored observations also addressed.

The rest of the paper is organized as follows. In Section 2, we propose an exact test for testing exponentiality against RIMRL$_{shock}$ class and then calculate the critical values for different sample sizes. The asymptotic normality proposed test statistic is proved in Section 3. The Pitman's asymptotic efficacy value is also given in this section. In Section 4, we report the result of simulation study carried out to assess the performance of the proposed test. In Section 5, we discuss how to incorporate right censored observation in our study.

2. **Exact Test**

Let X be the lifetime of a device which has absolutely continuous distribution function $F(.)$. Suppose $\bar{F}(x) = P(X > x)$ denotes the survival function of X at x. Also let $\mu = E(X) = \int_0^\infty \bar{F}(t)dt < \infty$. Assume that the device under consideration is experiencing a random shock. Suppose $N(t)$ denotes the total number of shocks up to time t with probability mass function $P(N(t) = j) = F^j(t) - F^{j+1}(t), j = 0, 1, 2, ...$ Suppose that the random variable $W_j, j = 0, 1, 2,$ quantify the amount of hidden lifetime absorbed by the jth shock with $W_0 = 0$ and having common distribution function $G(x) = P(W_j \leq x)$. The total cumulative life damage up to time t is defined as $Z(t) = \sum_{j=0}^{N(t)} W_j$ with the cumulative distribution function $Q(x) = P(Z(t) \leq x) =$

$\sum_{j=0}^{\infty} G^{(j)}[F^{(j)}(t) - F^{(j+1)}(t)]$. It is assumed that the unit fails when the total life-damage exceeds a pre-specified level $x > 0$. We refer to Glynn and Whitt (1993), Roginsky (1994) and Sepehrifar et al. (2015) for discussion related this framework.

Let $X^* = X - Z(t)$ be the residual lifetime of an operating device with cumulative damage $Z(t)$. Note that the realizations of X^* is available to us for further analysis. Consider a device subjected to $N(t)$ number of shocks up to time t. Given that such a device is in an operating situation at time instant t after installation, the MRL function of X^* denoted by $m^*(t)$ is defined by $m^*(t) = E(X^* - t|X^* \geq t)$. Note that the total life-damage will not exceed the threshold level x. From the definitions it is evident that the random variables X^* and $Z(t)$ are independent. Next we give the definition of RIMRL$_{shock}$ class (Sepehrifar et al., 2015).

Definition 2.1. *The mean residual life of a device under shock model (MRL$_{shock}$) at time t is defined as*

$$m^*(t) = \frac{1}{\bar{r}(t)} \int_t^{\infty} \bar{r}(z) dz,$$

where $\bar{r}(z) = \int_0^x \bar{F}(z + w) dQ(w)$.

Definition 2.2. *The random variable X belongs to the RIMRL$_{shock}$ class if the function $m^*(t)$ is a non-decreasing function for all $t > 0$.*

We are interested to test the null hypothesis

$$H_0 : F^* \quad \text{is exponential}$$

against the alternatives

$$H_1 : F^* \quad \text{is RIMRL}_{shock} \text{ (and not exponential)},$$

on the basis of a random sample $X_1^*, X_2^*, ..., X_n^*$; from an absolutely continuous distribution function F^*. For the above testing problem Sepehrifar et al. (2015) proposed a non-parametric test based on the departure measure $\Delta^*(F^*)$ defined by

$$\Delta^*(F^*) = \frac{1}{\mu^*} E_{f^*}(min(X_1^*, X_2^*) - \frac{1}{2}X_1^*) = \frac{\Delta(F^*)}{\mu^*},$$

where $\Delta(F^*) = E_{f^*}(min(X_1^*, X_2^*) - \frac{1}{2}X_1^*)$ and $\mu^* = E(X_1^*)$. Based on U-statistics theory Sepehrifar et al. (2015) obtained the following test statistic

$$\widehat{\Delta}^* = \frac{\widehat{\Delta}}{\bar{X}^*}, \tag{1}$$

where $\bar{X}^* = \frac{1}{n}\sum_{i=1}^n X_i^*$ and $\widehat{\Delta} = \frac{2}{n(n-1)}\sum_{i=1}^n \sum_{j<i;j=1}^n h(X_i^*, X_j^*)$ with $h(X_1^*, X_2^*) = min(X_1^*, X_2^*) - \frac{1}{2}X_1^*$. Hence the test procedure is to reject the null hypothesis H_0 in favour of H_1 for large values of $\widehat{\Delta}^*$.

Remark 2.1. *For the testing problem discussed here, Sepehrifar et al. (2015) obtained a critical region based on the asymptotic variance* $\frac{7}{48}$. *However as we shown (see Section 3) the asymptotic variance is* $\frac{1}{12}$. *We could not find enough details in their paper to explain the discrepancy.*

Motivated by this discrepancy, next we develop an exact test based on the test statistics $\widehat{\Delta}^*$ and calculate the critical values for different sample size. We use a result due to Box (1954) to find the exact null distribution of the test statistic.

Theorem 2.1. *Let X^* be continuous non-negative random variable with $\bar{F}^*(x) = e^{-\frac{x}{2}}$. Let $X_1^*, X_2^*, ..., X_n^*$ be independent and identical samples from F^*. Then for fixed n*

$$P(\widehat{\Delta}^* > x) = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \Big(\frac{d_{i,n} - x}{d_{i,n} - d_{j,n}}\Big) I(x, d_{i,n}),$$

provided $d_{i,n} \neq d_{j,n}$ for $i \neq j$, where

$$I(x,y) = \begin{cases} 1 & if \quad x \leq y \\ 0 & if \quad x > y \end{cases} \quad and \quad d_{i,n} = \frac{(n-2i+1)}{2(n-1)}.$$

Proof: First we express the test statistics in terms of order statistics. Note that

$$\frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} min(X_i^*, X_j^*) = \frac{2}{n(n-1)} \sum_{i=1}^{n} (n-i) X_i,$$

where $X_{(i)}^*$, $i = 1, 2, ..., n$, is the i-th order statistics based on the random sample $X_1^*, X_2^*, ...X_n^*$; from F^*. After some algebraic manipulation, we can express the equation (1) as

$$\widehat{\Delta}^* \quad = \quad \frac{\frac{1}{2n(n-1)} \sum_{i=1}^{n} (3n - 4i + 1) X_{(i)}^*}{\bar{X}^*}. \tag{2}$$

Rewrite the denominator of the equation (2) as

$$\widehat{\Delta} = \sum_{i=1}^{n} X_{(i)}^* \Big[\frac{(n-i+1)^2}{n(n-1)} - \frac{(n-i)^2}{n(n-1)} - \frac{(n+1)}{2n(n-1)}\Big].$$

Or

$$\widehat{\Delta} = \frac{n}{(n-1)} \sum_{i=1}^{n} X_{(i)}^* \Big[\frac{(n-i+1)^2}{n^2} - \frac{(n-i)^2}{n^2} - \frac{(n+1)}{2n^2}\Big].$$

Hence, in terms of the normalized spacings, $D_i = (n-i+1)(X^*_{(i)} - X^*_{(i-1)})$, with $X^*_0 = 0$, we can express the test statistics as

$$\widehat{\Delta}^* = \frac{\sum_{i=1}^{n} d_{i,n} D_i}{\sum_{i=1}^{n} D_i},$$

where $d_{i,n}$'s are given by

$$\begin{aligned} d_{i,n} &= \frac{1}{(n-1)}\big[(n-i+1) - \frac{(n+1)}{2}\big] \\ &= \frac{(n-2i+1)}{2(n-1)}. \end{aligned}$$

Note that the exponential random variable with rate $\frac{1}{2}$ is distritbuted same as the χ^2 random variable with 2 degrees of freedom. Hence the result follows from Theorem 2.4 of Box (1954).

The critical values of the exact test for different n are tabulated in Table 1.

TABLE 1. Critical values of the exact test

n	90% level	95% level	97.5% level	99% level
2	0.4000	0.4500	0.4750	0.4900
3	0.2764	0.3419	0.3883	0.4292
4	0.2189	0.2678	0.323	0.3693
5	0.1883	0.2383	0.28	0.325
6	0.1679	0.2131	0.2508	0.2927
7	0.1529	0.1944	0.2293	0.2682
8	0.1413	0.1799	0.2125	0.2492
9	0.1319	0.1682	0.1989	0.2336
10	0.1243	0.1586	0.1877	0.2208
15	0.0993	0.1271	0.1508	0.178
20	0.0852	0.109	0.1295	0.1531
25	0.0758	0.097	0.1153	0.1363
30	0.0689	0.0882	0.1049	0.1241
40	0.0594	0.0761	0.0905	0.1072
50	0.0529	0.0679	0.0808	0.0957
75	0.0431	0.0552	0.0658	0.078
100	0.0373	0.0477	0.0569	0.0675

3. Asymptotic properties

In this section, we prove asymptotic normality of the proposed test statistic. Making use of the asymptotic distribution we also calculate the Pitman's asymptotic efficacy of the test. As mentioned Sepehrifar et al. (2015) showed that the test statistic has limiting normal distribution, however they incorrectly stated the asymptotic variance. Hence we give the following results to correct the error occurred in their study.

Theorem 3.1. *The distribution of $\sqrt{n}(\widehat{\Delta} - \Delta(F^*))$, as $n \to \infty$, is Gaussian with mean zero and variance $4\sigma_1^2$, where σ_1^2 is the asymptotic variance of $\widehat{\Delta}$ and is given by*

$$\sigma_1^2 = \frac{1}{4} Var\Big(2X^* \bar{F^*}(X^*) + 2\int_0^{X^*} y dF^*(y) - \frac{1}{2}X^*\Big). \tag{3}$$

Corollary 3.1. *Let X^* be continuous non-negative random variable with $\bar{F^*}(x) = e^{-\frac{x}{\lambda}}$, then the distribution of $\sqrt{n}\widehat{\Delta}$, as $n \to \infty$, is Gaussian with mean zero and variance $\sigma_0^2 = \frac{\lambda^2}{12}$.*

Corollary 3.2. *Let X^* be continuous non-negative random variable with $\bar{F^*}(x) = e^{-\frac{x}{\lambda}}$, then the distribution of $\sqrt{n}\widehat{\Delta}^*$, as $n \to \infty$, is Gaussian with mean zero and variance $\sigma_0^2 = \frac{1}{12}$.*

Apart from the exact test we can construct an asymptotic test based on the asymptotic distribution of $\widehat{\Delta}^*$. Hence in case of the asymptotic test, for large values of n, we reject the null hypothesis H_0 in favour of

the alternative hypothesis H_1, if

$$\sqrt{12n}(\widehat{\Delta}^*) > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$. In fact, this is the correct critical region of the test proposed by Sepehrifar et al. (2015).

Next we study the asymptotic efficiency of the test. The Pitman's asymptotic efficacy is the most frequently used index to make a quantitative comparison of two distinct asymptotic tests for a certain statistical hypothesis. The Pitman's asymptotic efficacy (PAE) is defined as

$$PAE(\widehat{\Delta}^*) = \frac{|\frac{d}{d\lambda}\Delta^*(F^*)|_{\lambda\to\lambda_0}}{\sigma_0},$$

where λ_0 is the value of λ under H_0 and σ_0^2 is the asymptotic variance of $\widehat{\Delta}^*$ under H_0. In our case, the PAE is given by

$$\begin{aligned} PAE(\Delta^*(F^*)) &= \frac{|\frac{d}{d\lambda}\Delta^*(F^*)|_{\lambda\to\lambda_0}}{\sigma_0} \\ &= \sqrt{12}(W'(\lambda_0) - W(\lambda_0)\mu_a^{*'}(\lambda_0)), \end{aligned}$$

where $W = E(min(X_1^*, X_2^*))$ and μ_a^* is the mean of X^* under the alternative hypothesis and the prime denotes the differentiation with respect to λ. We calculate the PAE value for three commonly used alternatives which are the members of RIMRL$_{shock}$ class

(i) the Weibull family: $\bar{F}^*(x) = e^{-x^\lambda}$ for $\lambda > 1$, $x \geq 0$

(ii) the linear failure rate family: $\bar{F}^*(x) = e^{(-x-\frac{\lambda}{2}x^2)}$ for $\lambda > 0$, $x \geq 0$

(iii) the Makeham family: $\bar{F}^*(x) = e^{-x-\lambda(e^{-x}+x-1)}$ for $\lambda > 0$, $x \geq 0$.

By direct calculations, we observe that the PAE for Weibull distribution is equal to 1.2005; while for linear failure rate distribution and the Makeham distribution these values are, 0.8660 and 0.2828, respectively.

Next we compare the performance of the proposed test with some other tests available in the context of age replace model by evaluating the PAE of the respective tests. We compare our test with that tests proposed by Li and Xu (2008) and Kayid et al. (2013). The Table 2 gives the PAE values for different test procedures. From the Table 2, it is clear that our test is quite efficient for the Weibull and linear failure rate alternatives. Note that the test proposed by Kayid et al. (2013) has good efficacy for Makeham alternative even though their test shows poor performance against the other two given alternatives.

TABLE 2. Pitman's asymptotic efficacy (PAE)

Distribution	Proposed test	Li and Xu (2008)	Kayid et al. (2013)
Weibull	1.2005	1.1215	0.4822
Linear failure rate	0.8660	0.5032	0.4564
Makeham	0.2828	0.2414	2.084

4. **Simulation study**

Here we report a simulation study for evaluating the performance of our asymptotic test against various alternatives. The simulation was done using R program.

First we find the empirical type 1 error of the proposed test. Since the test is scale invariant, we simulate random sample from standard exponential distribution. The simulation is repeated for ten thousand times with different values of n and is reported in Table 3. From

Table 3. Empirical type 1 error of the test

n	Type 1 Error (5% level)	Type 1 Error (1% level)
10	0.0635	0.0123
20	0.0540	0.0115
30	0.0518	0.0107
40	0.0520	0.0110
50	0.0517	0.0107
60	0.0516	0.0102
70	0.0515	0.0102
80	0.0511	0.0100
90	0.0504	0.0103
100	0.0504	0.0104

the Table 3 it evident that the empirical type 1 error is a very good estimator of the size of the test even for small sample size.

For finding empirical power against various alternatives, we simulate observations from Weibull, linear failure rate and Makeham distributions with different values of λ where the distribution functions were given in the Section 3. The empirical powers for the above mentioned alternatives are given in Tables 4, 5 and 6. From these tables we can see that empirical powers of the test approaches one when the θ values are going away from the null hypothesis value as well as when n takes large values.

Table 4. Empirical Power: Weibull distribution

	$\lambda = 1.2$		$\lambda = 1.4$		$\lambda = 1.6$		$\lambda = 1.8$	
n	5%	1%	5%	1%	5%	1%	5%	1%
60	0.50	0.23	0.93	0.76	0.99	0.97	1.00	0.99
70	0.55	0.27	0.96	0.84	0.99	0.99	1.00	1.00
80	0.60	0.31	0.98	0.89	0.99	0.99	1.00	1.00
90	0.64	0.36	0.99	0.93	0.99	0.99	1.00	1.00
100	0.69	0.41	0.99	0.95	1.00	0.99	1.00	1.00

Table 5. Empirical Power: Linear failure rate distribution

	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
n	5%	1%	5%	1%	5%	1%	5%	1%
60	0.49	0.22	0.68	0.38	0.79	0.51	0.87	0.65
70	0.55	0.27	0.74	0.46	0.84	0.61	0.92	0.74
80	0.60	0.32	0.80	0.53	0.89	0.68	0.94	0.81
90	0.65	0.36	0.83	0.59	0.91	0.74	0.97	0.86
100	0.69	0.41	0.87	0.65	0.94	0.80	0.98	0.90

Table 6. Empirical Power: Makeham distribution

	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
n	5%	1%	5%	1%	5%	1%	5%	1%
60	0.37	0.14	0.49	0.22	0.65	0.36	0.87	0.63
70	0.42	0.17	0.55	0.26	0.72	0.43	0.92	0.72
80	0.46	0.20	0.60	0.31	0.78	0.49	0.94	0.79
90	0.51	0.23	0.65	0.35	0.82	0.56	0.96	0.84
100	0.55	0.27	0.70	0.40	0.86	0.62	0.98	0.90

5. **The case of censored observations**

Next we discuss how the censored observations can be incorporated in the proposed method. Suppose we have randomly right-censored observations such that the censoring times are independent of the life-times. Under this set up the observed data are n independent and identical copies of (Y^*, δ), with $Y^* = min(X^*, C)$, where C is the censoring time and $\delta = I(X^* \leq C)$. Now we need to address the testing problem mentioned in Section 2 based on n independent and identical observation $\{(Y_i^*, \delta_i), 1 \leq i \leq n\}$. Observe that $\delta_i = 1$ means i^{th} object is not censored, whereas $\delta_i = 0$ means that i^{th} object is censored by C, on the right. Usually we need to redefine the measure $\Delta^*(F^*)$ to incorporates the censored observations. The U-statistics formulations helps us to solve the problem in an easy way. Using the right-censored

version of a U-statistic introduced by Datta et al. (2010) an estimator $\Delta(F^*)$ with censored observation is given by

$$\widehat{\Delta}_c = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} \frac{h(Y_i^*, Y_j^*)\delta_i\delta_j}{\widehat{K}_c(Y_i^*)\widehat{K}_c(Y_j^*)},$$

where $h(Y_1^*, Y_2^*) = \frac{1}{4}(4Y_1^* I(Y_1^* < Y_2^*) + 4Y_2^* I(Y_2^* < Y_1^*) - Y_1^* - Y_2^*)$, provided $\widehat{K}_c(Y_i^*), \widehat{K}_c(Y_j^*) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of the censoring variable C. Similarly an estimator of μ^* is given by (Zhao and Tsiatis, 2000)

$$\widehat{X}_c^* = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i^* \delta_i}{\widehat{K}_c(Y_i^*)}.$$

Hence in right censoring situation, the test statistics is given by

$$\widehat{\Delta}_c^* = \frac{\widehat{\Delta}_c}{\widehat{X}_c^*},$$

and the test procedure is to reject H_0 in favour of H_1 for large values of $\widehat{\Delta}_c^*$.

Next we obtain the limiting distribution of the test statistic. Let $N_i^c(t) = I(Y_i^* \le t, \delta_i = 0)$ be the counting process corresponds to the censoring variable for the i^{th} individual, $Z_i(t) = I(Y_i^* \ge t)$. Also let λ_c be the hazard rate of C. The martingale associated with this counting process is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t Z_i(u)\lambda_c(u)du.$$

Let $G(x,y) = P(X_1^* \le x, Y_1^* \le y, \delta = 1), x \in \mathcal{X}$, $H(t) = P(Y_1^* \le t)$ and

$$w(t) = \frac{1}{\bar{H}(t)} \int_{\mathcal{X}\times[0,\infty)} \frac{h_1(x)}{K_c(y-)} I(y > t) dG(x,y),$$

where $h_1(x) = Eh(x, X_2^*)$. Next result follows from Datta et al. (2010) for the choice of the kernel $h(Y_1^*, Y_2^*) = \frac{1}{4}(4Y_1^* I(Y_1^* < Y_2^*) + 4Y_2^* I(Y_2^* < Y_1^*) - Y_1^* - Y_2^*)$.

Theorem 5.1. *If $Eh^2(Y_1^*, Y_2^*) < \infty$, $\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x,y) < \infty$ and $\int_0^\infty w^2(t)\lambda_c(t)dt < \infty$, then the distribution of $\sqrt{n}(\widehat{\Delta}_c - \Delta(F^*))$, as $n \to \infty$, is Gaussian with mean zero and variance $4\sigma_{1c}^2$, where σ_{1c}^2 is given by*

$$\sigma_{1c}^2 = Var\Big(\frac{h_1(X^*)\delta_1}{K_c(Y_1^*-)} + \int w(t) dM_1^c(t)\Big).$$

Corollary 5.1. *Under the assumptions of Theorem 5.1, if $E(Y_1^2) < \infty$, the distribution of $\sqrt{n}(\widehat{\Delta}_c^* - \Delta^*(F^*))$, as $n \to \infty$, is Gaussian with mean zero and variance $4\sigma_c^2$, where*

$$\sigma_c^2 = \frac{\sigma_{1c}^2}{\mu^{*2}}. \tag{4}$$

Corollary 5.2. *Let X^* be continuous non-negative random variable with $\bar{F}^*(x) = e^{-\frac{x}{\lambda}}$. Under the assumptions of Theorem 5.1, if $E(Y_1^2) < \infty$, the distribution of $\sqrt{n}\widehat{\Delta}_c^*$, as $n \to \infty$, is Gaussian with mean zero and variance σ_{c0}^2, where*

$$\sigma_{c0}^2 = \frac{4}{\lambda^2} Var\Big(\frac{(4F^*(X^*) - X^*)\delta_1}{4K_c(Y_1^*-)} + \int w(t) dM_1^c(t)\Big). \tag{5}$$

Hence by Corollary 5.2, we know that the $\sqrt{n}\widehat{\Delta}_c^*$ has asymptotically normal with mean zero and a variance that can be estimated by (5)

and we denote it as $\widehat{\sigma}_{c0}$. Hence we reject the null hypothesis in favour of H_1, if

$$\frac{\sqrt{n}\widehat{\Delta}_c^*}{\widehat{\sigma}_{c0}} \geq Z_\alpha.$$

Next we study the efficiency loss due to censoring by computing the efficiency of our test based on $\widehat{\Delta}^*$ for uncensored model and the efficiency of the test based on $\widehat{\Delta}_c^*$ for censored model. As both these tests have same asymptotic mean, the Pitman asymptotic relative efficiency (ARE) of the test based on $\widehat{\Delta}_c^*$ with respect to the test based on $\widehat{\Delta}^*$ is given by

$$e = ARE(\widehat{\Delta}_c^*, \widehat{\Delta}^*) = \frac{\sigma_0^2}{\sigma_{c0}^2}.$$

The quantity $(1-e)$ can be taken as a measure of the efficiency loss (Lim and Park, 1993) due to censoring. From the above expression it is clear that the ARE value is independent of the distributions belonging to the family of alternative hypothesis, but depends on the distribution of C. Next we calculate the ARE value when the censoring variable C

Table 7. Asymptotic relative efficiency

λ	0.5	0.4	0.3	0.2	0.1	0.05	0.01
ARE	0.397	0.433	0.480	0.547	0.643	0.700	0.741

has logistic distribution with distribution function $F(x) = \frac{1}{1+e^{-\frac{x}{\lambda}}}$. The ARE value for different values of λ is given in Table 7. Table 7 shows that as λ decreases, the value of ARE increases and the efficiency loss decreases as the value of λ (the amount of censoring) becomes small.

References

[1] Box, G. E. P. (1954). Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification, *Annals of Mathematical Statistics,* 25, 290-302.

[2] Datta, S., Bandyopadhyay, D. and Satten, G. A. (2010). Inverse probability of censoring weighted U-statistics for right-censored data with an application to testing hypotheses. *Scandinavian Journal of Statistics*, 37, 680-700.

[3] Glynn, P.W. and Whitt, W. (1993). Limit theorems for cumulative processes, *Stochastic Processes and their Applications*, 47, 299-314

[4] Kayid, M., Ahmad, I.A., Izadkhah S. and Abouammoh, A.M. (2013). Further results involving the mean time to failure order, and the decreasing mean time to failure class, *IEEE Transactions on Reliability*, 62, 670-678.

[5] Li, X and Xu, M. (2008). Reversed hazard rate order of equilibrium distributions and a related ageing notion, *Statistical Papers*, 49,749-767.

[6] Lim, J.H and Park, D.H. (1993). Test for DMRL using censored data, *Journal of Nonparametric statistics*, 3, 167- 173.

[7] Roginsky, A.L (1994). A central limit theorem for cumulative processes, *Advances in Applied Probability*, 26, 104-121.

[8] Sepehrifar, M.B, Khorshidian, K. and Jamshidian, A.R. (2015). On renewal increasing mean residual life distributions: An age replacement model with hypothesis testing application, *Statistics and Probability*, 96, 117-122.

[9] Zhao, H. and Tsiatis, A. A. (2000). Estimating Mean Quality Adjusted Lifetime with Censored Data. *Sankhya-B,* 62, 175-188.